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LETTER TO THE EDITOR

Functional connection between BG scattering amplitude and Glauber scattering amplitude

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Abstract. The functional connection between Blankenbecler–Goldberger (BG) amplitude and Glauber amplitude have been found, so that a fast way to reduce the integral expression of BG amplitude in the scattering problem can be acquired. Two equivalent expressions are discussed.

The Blankenbecler–Goldberger (1962) approximation was proposed for discussing the scattering problem of high energy. The essence of this approximation is a modification of the Green function eikonalization in Glauber approximation (GA), i.e. the BG approximation requires that one retains only the on shell part after the eikonalization of the propagator (Kamal 1972). In theory, the BG approximation is consistent with dispersion theory (Blankenbecler and Goldberger 1962).

As is well known, the Glauber approximation has been extensively used in various collision problems at high energy, and there are many methods and calculating skills for reducing the integral expression of Glauber scattering amplitude (Chan *et al* 1979, Gien 1988). The reduction of BG amplitude, however, is more difficult than the Glauber amplitude. In this letter, the functional connection between BG amplitude and Glauber amplitude has been found, so that a fast way to reduce BG amplitude in the scattering problem has been acquired.

In order for the reader to understand the nature of the results presented here, it is useful to briefly give the derivation of BG amplitude in potential scattering. In a previous paper (Chen *et al* 1992), the BG amplitude and BG wavefunction have been derived from dispersion theory following Goldberger (1962, 1964). Now another method is given which follows Kamal (1972) and provides a simple means to obtain BG amplitude. According to the standard integral equation for the T-matrix

$$T = V + T\frac{1}{E + \mathrm{i}\varepsilon - H_0}V$$

the scattering amplitude is (Goldberger and Watson 1964)

$$f(\boldsymbol{K}, \boldsymbol{K}') = f_B(\boldsymbol{K}, \boldsymbol{K}') + \int \frac{\mathrm{d}q^3}{(2\pi)^3}\frac{U(\boldsymbol{K}-\boldsymbol{q})f(\boldsymbol{q}, \boldsymbol{K}')}{K^2 + \mathrm{i}\varepsilon - q^2} \tag{1}$$

where $f_B(\boldsymbol{K}, \boldsymbol{K}') = -(1/4\pi)U(\boldsymbol{K}-\boldsymbol{K}')$, and $U = 2mV/\hbar^2 = 2V(h = c = m = 1)$.

Let us introduce the wavevector transfer $\boldsymbol{K}-\boldsymbol{K}' = \boldsymbol{\lambda}$, $\boldsymbol{K}-\boldsymbol{q} = \boldsymbol{\xi}$ and $\boldsymbol{q}-\boldsymbol{K}' = \boldsymbol{\lambda}-\boldsymbol{\xi} \equiv \boldsymbol{\eta}$. Then equation (1) can be written as

$$f(K, K', \boldsymbol{\lambda}) = f_B(\boldsymbol{\lambda}) - \int \frac{\mathrm{d}^3\xi}{(2\pi)^3}\frac{U(\boldsymbol{\xi})f(K, K', \xi, \boldsymbol{\eta})}{(\boldsymbol{K}-\boldsymbol{\xi})^2 - K^2 - \mathrm{i}\varepsilon}. \tag{2}$$

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After eikonalization of the propagator and for the small momentum transfer ξ^2, equation (2) becomes

$$f(K, K', \boldsymbol{\lambda}) = f_B(\boldsymbol{\lambda}) + \int \frac{\mathrm{d}^3\xi}{(2\pi)^3} \frac{U(\xi) f(K, K', \boldsymbol{\eta})}{2K\xi_z + \mathrm{i}\varepsilon}. \tag{3}$$

We define two new functions H and B as follows:

$$H(K, K', b, \lambda_z) = \frac{1}{2\pi} \int f(K, K', \boldsymbol{\lambda}) \exp[-\mathrm{i}\boldsymbol{\lambda}_\perp \cdot \boldsymbol{b}] \, \mathrm{d}^2\lambda_\perp \tag{4a}$$

$$B(b, \lambda_z) = \frac{1}{2\pi} \int U(\boldsymbol{\lambda}) \exp[-\mathrm{i}\boldsymbol{\lambda}_\perp \cdot \boldsymbol{b}] \, \mathrm{d}^2\lambda_\perp. \tag{4b}$$

The inverse transform is

$$f(K, K', \boldsymbol{\lambda}) = \frac{1}{2\pi} \int H(K, K', b, \lambda_z) \exp[\mathrm{i}\boldsymbol{\lambda}_\perp \cdot \boldsymbol{b}] \, \mathrm{d}^2 b \tag{5a}$$

$$U(\boldsymbol{\lambda}) = \frac{1}{2\pi} \int B(b, \lambda_z) \exp[\mathrm{i}\boldsymbol{\lambda}_\perp \cdot \boldsymbol{b}] \, \mathrm{d}^2 b. \tag{5b}$$

From equation (3) we can obtain

$$H(K, K', b, \lambda_z) + \frac{1}{4\pi} b(b, \lambda_z) = \int \frac{\mathrm{d}\xi_z \, B(b, \xi_z) H(K, K', b, \eta_z)}{(2\pi)^2 (2K\xi_z + \mathrm{i}\varepsilon)}. \tag{6}$$

For on-energy-shell process, $1/(\xi_z + \mathrm{i}\varepsilon) \rightarrow -\mathrm{i}\pi\delta(\xi_z)$ (Kamal and Chavda 1972). Therefore from equation (6) we have

$$H(K, K', b, \lambda_z) = \frac{-B(b, \lambda_z)/4\pi}{1 + \mathrm{i}B(b)/8\pi K}. \tag{7}$$

Using equations (4*a*)–(5*b*) and noting that $\boldsymbol{\lambda} \cdot \boldsymbol{b} = \boldsymbol{\lambda}_\perp \cdot \boldsymbol{b}$ for small scattering angle, we can obtain

$$B(b) = 2\pi \int U[(b^2 + Z^2)^{1/2}] \, \mathrm{d}Z$$

$$f(K, K', \boldsymbol{\lambda}) = \frac{K}{2\pi} \int \mathrm{d}^2 b \, \mathrm{e}^{\mathrm{i}\boldsymbol{\lambda}\cdot\boldsymbol{b}} \frac{X(b, K)}{1 - \mathrm{i}X(b, K)/2}. \tag{8}$$

This is just the BG amplitude expression potential scattering, where $X(b, K)$ is the eikonal phase

$$X(b, K) = -(1/K) \int_0^\infty U[(b^2 + Z^2)^{1/2}] \, \mathrm{d}Z. \tag{9}$$

The wavefunction corresponding to the BG amplitude is

$$\Psi^{\mathrm{BG}} = \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r}} \{1 + (\mathrm{i}/4K) \int_{-\infty}^{z} \mathrm{d}Z' \, U[(b^2 + Z'^2)^{1/2}]\}^{-2}. \tag{10}$$

This is already known. If we take Ψ^{BG} in equation (10) as ψ^+ and use the standard formation of the scattering amplitude

$$f = -(1/4\pi) \langle \varphi_f | U | \psi^+ \rangle \tag{11}$$

we can immediately get the result of equation (8).

For the real potential function U, we can use the Laplace transform

$$\alpha^{-n}=\frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}\,\mathrm{e}^{-\alpha t}\,\mathrm{d}t \qquad (\mathrm{Re}(\alpha)>0) \tag{12}$$

so that we have

$$\left[1+(\mathrm{i}/4K)\int_{-\infty}^{Z}\mathrm{d}Z'\,U(b,Z')\right]^{-2}=\int_0^\infty t\,\mathrm{d}t\,\mathrm{e}^{-t}\exp\left(-\frac{\mathrm{i}t}{4K}\int_{-\infty}^{Z}\mathrm{d}Z'\,U(b,Z')\right). \tag{13}$$

As is well known, the Glauber wavefunction is

$$\Psi^{\mathrm{GA}}(\boldsymbol{r},\xi)=\exp\left(\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{r}-\mathrm{i}\xi\int_{-\infty}^{Z}\mathrm{d}Z'\,U(b,Z')\right) \tag{14}$$

where $\xi=\frac{1}{2}K$. Combining equations (10), (13) and (14), we can obtain

$$\Psi^{\mathrm{BG}}(\boldsymbol{r})=\int_0^\infty t\,\mathrm{d}t\,\mathrm{e}^{-t}\,\Psi^{\mathrm{GA}}(\boldsymbol{r},\xi t/2). \tag{15}$$

Substituting equation (15) into equation (11), the functional relation between BG amplitude and Glauber amplitude can be obtained as follows:

$$f_{fi}^{\mathrm{BG}}=\int_0^\infty t\,\mathrm{d}t\,\mathrm{e}^{-t}f_{fi}^{\mathrm{GA}}(\xi t/2) \tag{16}$$

where

$$\begin{aligned} f_{fi}^{\mathrm{Ga}}(\xi t/2)&=-(1/4\pi)\langle\varphi_f|U|\Psi^{\mathrm{GA}}(\boldsymbol{r},\xi t/2)\rangle\\ &=-(1/4\pi)\int\mathrm{d}\boldsymbol{r}\exp\mathrm{i}(\boldsymbol{K}-\boldsymbol{K}')\cdot\boldsymbol{r}U(r)\\ &\quad\times\exp\left(-\mathrm{i}(\xi t/2)\int_{-\infty}^{Z}\mathrm{d}Z'\,U(b,Z')\right)\\ &\simeq(\mathrm{i}/4\pi)(2/\xi t)\int\mathrm{d}^2b\,\mathrm{e}^{\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}(1-\mathrm{e}^{\mathrm{i}Xt/2}). \end{aligned} \tag{17}$$

If we note that

$$x/(1-\mathrm{i}x/2)=4\mathrm{i}\int_0^\infty\mathrm{d}t(1-\mathrm{e}^{\mathrm{i}Xt})\,\mathrm{e}^{-2t} \tag{18}$$

and substitute this into equation (8), we can obtain another formation

$$f_{fi}^{\mathrm{BG}}=4\int_0^\infty\mathrm{d}t\,\mathrm{e}^{-2t}\,F_{fi}^{\mathrm{GA}}(\xi t) \tag{19}$$

where

$$F_{fi}^{\mathrm{GA}}(\xi t)=(\mathrm{i}K/2\pi)\int\mathrm{d}^2b\,\mathrm{e}^{\mathrm{i}\boldsymbol{\Delta}\cdot\boldsymbol{b}}(1-\mathrm{e}^{\mathrm{i}Xt}). \tag{20}$$

It is noteworthy that f_{fi}^{GA} of equation (16) is not the same as F_{fi}^{GA} of equation (19), but equations (16) and (19) are equivalent. In order to understand this statement, we take equation (17) into equation (16) and change integrating variable t to $2t$ so that we have

$$f_{fi}^{\mathrm{BG}} = 2(\mathrm{i}K/2\pi)\int \mathrm{d}^2b\,\mathrm{e}^{\mathrm{i}\lambda\cdot b}(1-\mathrm{e}^{\mathrm{i}\chi t/2})\,\mathrm{e}^{-t}\,\mathrm{d}t$$

$$= 4(\mathrm{i}K/2\pi)\int \mathrm{d}^2b\,\mathrm{e}^{\mathrm{i}\lambda\cdot b}(1-\mathrm{e}^{\mathrm{i}\chi t})\,\mathrm{e}^{-2t}\,\mathrm{d}t$$

which is just equation (19).

It is important to note the difference between f_{fi}^{GA} and F_{fi}^{GA} in the practical application of equations (16) and (19). As an example, we employ the BG amplitude to extend to the calculation of e–H elastic scattering. If equation (20) is taken as the Glauber amplitude, the e–H elastic scattering amplitude is (Kamal *et al* 1976, Thomas and Gerjuoy 1971)

$$F_{fi}^{\mathrm{GA}}(\xi) = (\mathrm{i}K/2\pi)\int u_f^*(r)(1-\mathrm{e}^{\mathrm{i}\chi})u_i^*(r)\,\mathrm{e}^{\mathrm{i}q\cdot b}\,\mathrm{d}^3r\,\mathrm{d}^2b$$

$$= \mathrm{i}K\frac{(2\mathrm{i}\zeta)}{\lambda^2}\Gamma(1+\mathrm{i}\zeta)\Gamma(1-\mathrm{i}\zeta)(2/q)^{2-2\mathrm{i}\zeta}$$

$$\times[(1+\mathrm{i}\zeta)\,{}_2F_1(1-\mathrm{i}\zeta, 1-\mathrm{i}\zeta; 1; 4a_0^2/q^2)$$

$$+(4a_0^2/q^2)(1-\mathrm{i}\zeta)^2\,{}_2F_1(2-\mathrm{i}\zeta, 2-\mathrm{i}\zeta; 4a_0^2/q^2)] \qquad (21)$$

where a_0 is the first Bohr radius, $q = K - K'$ is the momentum transfer, $\lambda = 2/a_0$ and $\zeta = 2\xi$. We take ξt instead of ξ in the expression (21) and then substitute it into the expression (19). We can immediately obtain the BG amplitude which only includes one-dimension integration.

If equation (17) is taken as the Glauber amplitude, we then have

$$f_{fi}^{\mathrm{GA}}(\xi t/2) = (2\mathrm{i}K/t)(2\mathrm{i}\rho/\lambda^2)\Gamma(1+\mathrm{i}\rho)\Gamma(1-\mathrm{i}\rho)(2/q)^{2-2\mathrm{i}\rho}$$

$$\times[(1+\mathrm{i}\rho)\,{}_2F_1(1-\mathrm{i}\rho, 1-\mathrm{i}\rho; 1; 4a_0^2/q^2)$$

$$+(4a_0^2/q^2)(1-\mathrm{i}\rho)^2\,{}_2F_1(2-\mathrm{i}\rho, 2-\mathrm{i}\rho; 2; 4a_0^2/q^2) \qquad (22)$$

where $\rho = 2(\xi t/2)$. It is important to note that we have provided a better method to reduce the integral expression of the BG amplitude in the collision problem using equations (16) and (19).

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